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FIXED PARAMETER ALGORITHMS FOR RESTRICTED COLORING PROBLEMS

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Abstract. In this paper, we obtain polynomial time algorithms to determine the acyclic chromatic number, the star chromatic number, the Thue chromatic number, the harmonious chromatic number and the clique chromatic number of P_4-tidy graphs and $(q, q-4)$-graphs, for every fixed q. These classes include cographs, P_4-sparse and P_4-lite graphs. All these coloring problems are known to be NP-hard for general graphs. These algorithms are fixed parameter tractable on the parameter $q(G)$, which is the minimum q such that G is a $(q, q-4)$-graph. We also prove that every connected $(q, q-4)$-graph with at least q vertices is 2-clique-colorable and that every acyclic coloring of a cograph is also nonrepetitive.

1. Introduction

Let $G = (V, E)$ be a finite undirected graph, without loops and multiple edges. The complete bipartite graph with partitions of size m and n is denoted by $K_{m,n}$. A $K_{1,n}$ is called a star. A P_4 is an induced path with four vertices. A cograph is any P_4-free graph. The graph terminology used here follows [11].

A k-coloring of G is a partition $\{V_1, \dots, V_k\}$ of $V(G)$. The subsets $V_1, \dots, V_k$ are called *color classes* and we say that a vertex in V_i is colored i. A proper k-coloring is a k-coloring such that every color class induces a stable set. The chromatic number $\chi(G)$ of G is the smallest integer k such that G admits a proper k-coloring.

An *acyclic* coloring is a proper coloring such that every cycle receives at least three colors (that is, every pair of color classes induces a forest). A *star* coloring is an acyclic coloring such that every P_4 receives at least three colors (that is, every pair of color classes induces a forest of stars). A *nonrepetitive* coloring is a star coloring such that no path has a xx pattern of colors, where x is a sequence of colors. A *harmonious* coloring is a nonrepetitive coloring such that every pair of color classes induces at most one edge.

It is easy to see that any coloring of a split or chordal graph is acyclic. In 2011, Lyons [23] proved that every acyclic coloring of a cograph is also a star coloring. In this paper, we prove that it is also nonrepetitive.

The acyclic, star, Thue and harmonious chromatic numbers of G, denoted respectively by $\chi_a(G)$, $\chi_{st}(G)$, $\pi(G)$, $\chi_h(G)$, are the minimum number of colors k such that G admits an acyclic, star, nonrepetitive and harmonious coloring with k colors. By definitions,

$$\chi(G) \;\leq\; \chi_a(G) \;\leq\; \chi_{st}(G) \;\leq\; \pi(G) \;\leq\; \chi_h(G).$$

Determining the acyclic chromatic number is NP-Hard even for bipartite graphs [14] and deciding if $\chi_a(G) \leq 3$ is NP-Complete [21]. In 2004, Albertson et al. [3] proved that computing the star chromatic number is NP-hard even for planar bipartite graphs. In 2007, Asdre et al. [4] proved that determining the harmonious chromatic number is NP-hard for interval graphs, permutation graphs and split graphs.

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Borodin proved that $\chi_a(G) \leq 5$ for every planar graph G [12]. In 2004, Fertin, Raspaud and Reed give exact values of $\chi_{st}(G)$ for several graph classes [15]. In 2004, Campbell and Edwards [13] obtained new lower bounds for $\chi_h(G)$ in terms of the independence number.

In 2002, Alon et al. [1] proved a relation between the $\pi(G)$ and $\Delta(G)$. In 2008, Barát and Wood [9] proved that every graph G with treewidth t and maximum degree Δ satisfies $\pi(G) = O(k\Delta)$ (it was also proved that $\pi(G) \leq 4^t$ [18]). In 2009, Marx and Schaefer [24] proved that determining whether a particular coloring of a graph is nonrepetitive is coNP-hard, even if the number of colors is limited to four. In 2010, Grytczuk et al. [20] investigated list colorings which are nonrepetitive and proved that the Thue choice number of P_n is at most 4 for every n. See [18] and [19] for a survey on nonrepetitive colorings.

A *clique* coloring is a coloring (not necessarily a proper coloring) such that every maximal clique receives at least two colors. The clique chromatic number $\chi_c(G)$ is the minimum number k such that G has a clique coloring with k colors.

In 2002, Kratochvíl and Tuza [22] proved that determining the clique-chromatic number is polynomial time solvable for planar graphs, but is NP-Hard for perfect graphs. In 2004, Bacsó et al. [2] proved several results for 2-clique-colorable graphs.

Many NP-hard problems were proved to be polynomial time solvable for cographs. For example, Lyons [23] obtained a polynomial time algorithm to find an optimal acyclic and an optimal star coloring of a cograph. However, it is known that computing the harmonious chromatic number of a disconnected cograph is NP-hard [10].

Some superclasses of cographs, defined in terms of the number and structure of its induced P_4's, can be completely characterized by their modular or primeval decomposition. Among these classes, we cite P_4-sparse graphs, P_4-lite graphs, P_4-tidy graphs and $(q, q-4)$-graphs.

Babel and Olariu [7] defined a graph as $(q, q-4)$-*graph* if no set of at most q vertices induces more than $q-4$ distinct P_4's. Cographs and P_4-sparse graphs are precisely $(4, 0)$-graphs and $(5, 1)$-graphs respectively. P_4-lite graphs are special $(7, 3)$-graphs. We say that a graph is P_4-tidy if, for every P_4 induced by $\{u, v, x, y\}$, there exists at most one vertex z such that $\{u, v, x, y, z\}$ induces more than one P_4. Since the complement of a P_4 is also a P_4, these graph classes are closed under complementation.

In this paper, we prove the following result:

Theorem 1.1 (main theorem). *Let q be a fixed integer and let G be a P_4-tidy or a $(q, q-4)$-graph. There exists linear time algorithms to obtain*

- *$\chi_a(G)$, $\chi_{st}(G)$, $\pi(G)$ and $\chi_c(G)$;*
- *$\chi_h(G)$, if G is also connected.*

Moreover, every connected $(q, q-4)$-graph with at least q vertices is 2-clique-colorable, and every acyclic coloring of a cograph is also nonrepetitive.

Let $q(G)$ be the minimum integer q such that G is a $(q, q-4)$-graph. Theorem 1.1 proves that the acyclic, the star, the nonrepetitive, the harmonious and the clique coloring problems are fixed parameter tractable on the parameter $q(G)$.

2. Primeval and Modular decompositions

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex disjoint graphs. The disjoint union of G_1 and G_2 is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The join is the graph $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{uv : \ u \in V_1, \ v \in V_2\})$.

A *spider* is a graph whose vertex set has a partition (R, C, S), where $C = \{c_1, \ldots, c_k\}$ and $S = \{s_1, \ldots, s_k\}$ for $k \geq 2$ are respectively a clique and a stable set; s_i is adjacent to c_j if and only if $i = j$ (a thin spider), or s_i is adjacent to c_j if and only if $i \neq j$ (a thick spider); and every vertex of R is adjacent to each vertex of C and non-adjacent to each vertex of S.

Jamison and Olariu [7] proved an important structural theorem for $(q, q-4)$-graphs, using their primeval decomposition, which can be obtained in linear time. A graph is *p-connected* if, for every bipartition of the vertex set, there is a crossing P_4. A *separable p-component* is a maximal p-connected subgraph with a particular bipartition (H_1, H_2) such that every crossing P_4 $wxyz$ satisfies $x, y \in H_1$ and $w, z \in H_2$.

Theorem 2.1 (Characterizing $(q, q-4)$-graphs [7])**.** *A graph G is a $(q, q-4)$-graph if and only if exactly one of the following holds:*

(a) *G is the union or the join of two $(q, q-4)$-graphs;*
(b) *G is a spider (R, C, S) and $G[R]$ is a $(q, q-4)$-graph;*
(c) *G contains a separable p-component H, with bipartition (H_1, H_2) and $|V(H)| \leq q$, such that $G-H$ is a $(q, q-4)$-graph and every vertex of $G-H$ is adjacent to every vertex of H_1 and non-adjacent to every vertex of H_2;*
(d) *G has at most q vertices or $V(G) = \emptyset$.*

Using the modular decomposition of P_4-tidy graphs, Giakoumakis et al. proved a similar result for this class [17]. A *quasi-spider* is a graph obtained from a spider (R, C, S) by replacing at most one vertex from $C \cup S$ by a K_2 (the complete graph on two vertices) or a $\overline{K_2}$ (the complement of K_2).

Theorem 2.2 (Characterizing P_4-tidy graphs [17])**.** *A graph G is a P_4-tidy graph if and only if exactly one of the following holds:*

(a) *G is the union or the join of two P_4-tidy graphs;*
(b) *G is a quasi-spider (R, C, S) and $G[R]$ is a P_4-tidy graph;*
(c) *G is isomorphic to P_5, $\overline{P_5}$, C_5, K_1 or $V(G) = \emptyset$.*

As a consequence, a $(q, q-4)$-graph (resp. a P_4-tidy graph) G can be decomposed by successively applying Theorem 2.1 (resp. Theorem 2.2) as follows: If (a) holds, apply the theorem to each component of G or $\overline{G}$ (operations disjoint union and join). If (b) holds, apply the theorem to $G[R]$ (operation spider or quasi-spider). Finally, if (c) holds and G is a $(q, q-4)$-graph, then apply the theorem to $G-H$ (operation small subgraph).

It was also proved in [7] that every p-connected $(q, q-4)$-graph with $q \geq 8$ has at most q vertices. With this, we can obtain $q(G)$ in $O(n^7)$ time for every graph G from its primeval decomposition (observe that $q(G)$ can be greater than n and, if this is the case, $q(G)$ is the number of induced P_4's of G plus four).

The idea now is to consider the graph by the means of its decomposition tree obtained as described. According to the coloring parameter to be determined, the tree will be visited in an up way or bottom way fashion. We notice that the primeval and modular decomposition of any graph can be obtained in linear time [7].

3. Disjoint Union, Join and Spiders

We start by recalling a result from [23] for the acyclic and the star chromatic numbers.

Lemma 3.1 (χ_a and χ_{st} for union and join [23])**.** *Given graphs G_1 and G_2 with n_1 and n_2 vertices res-pectively:*

$$\chi_a(G_1 \cup G_2) = \max\{\chi_a(G_1), \chi_a(G_2)\},$$

$$\chi_{st}(G_1 \cup G_2) = \max\{\chi_{st}(G_1), \chi_{st}(G_2)\},$$

$$\chi_a(G_1 \vee G_2) = \min\{\chi_a(G_1) + n_2, \chi_a(G_2) + n_1\},$$

$$\chi_{st}(G_1 \vee G_2) = \min\{\chi_{st}(G_1) + n_2, \chi_{st}(G_2) + n_1\}.$$

The next lemma shows how to obtain the Thue chromatic number for union and join operations. It is easy to see that Lemmas 3.1 and 3.2 implies that, if G is a cograph, then $\pi(G) = \chi_{st}(G) = \chi_a(G)$ and every acyclic coloring of a cograph is also nonrepetitive.

Lemma 3.2 ($\pi(G)$ for union and join)**.** *Given graphs G_1 and G_2 with n_1 and n_2 vertices res-pectively:*

$$\pi(G_1 \cup G_2) = \max\{\pi(G_1), \pi(G_2)\},$$
$$\pi(G_1 \vee G_2) = \min\{\pi(G_1) + n_2, \pi(G_2) + n_1\}.$$

The two following lemmas deal with spiders and quasi-spiders and are proved in Section 5. We will consider $\chi_a(G[R]) = \chi_{st}(G[R]) = 0$ whenever $R = \emptyset$.

Lemma 3.3 (χ_a and χ_{st} for spiders)**.** *Let G be a spider (R, C, S), where $|C| = |S| = k$. Then $\chi_a(G) = \chi_a(G[R]) + k$ and $\chi_{st}(G) = \chi_{st}(G[R]) + k$, unless $R = \emptyset$ and G is thick, when in this case, $\chi_{st}(G) = k + 1$. Moreover, $\pi(G) = \chi_{st}(G)$.*

Lemma 3.4 (χ_a and χ_{st} for quasi-spiders)**.** *Let G be a quasi-spider (R, C, S) such that $\min\{|C|, |S|\} = k$ and $\max\{|C|, |S|\} = k + 1$. Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$. Then*

$$\chi_a(G) = \begin{cases} \chi_a(G[R]) + k + 1, & \text{if } H \in C, \\ \chi_a(G[R]) + k + 1, & \text{if } H = K_2,\ G \text{ is thick and } R = \emptyset, \\ \chi_a(G[R]) + k, & \text{otherwise,} \end{cases}$$

$$\chi_{st}(G) = \begin{cases} \chi_{st}(G[R]) + k, & \text{if } H \in S \text{ and } G \text{ is thin,} \\ \chi_{st}(G[R]) + k, & \text{if } H \in S,\ G \text{ is thick and } R \neq \emptyset, \\ \chi_{st}(G[R]) + k + 2, & \text{if } H \in C,\ G \text{ is thick and } R = \emptyset, \\ \chi_{st}(G[R]) + k + 1, & \text{otherwise.} \end{cases}$$

Moreover, $\pi(G) = \chi_{st}(G)$.

Lemma below determines the harmonious chromatic number for join and spider operations. Recall that χ_h for union operation is NP-hard [10].

Lemma 3.5 (χ_h for join and quasi-spiders)**.** *Let G be a graph with n vertices. If G is the join of two graphs G_1 and G_2, then $\chi_h(G) = n$. If G is a quasi-spider (R, C, S) with $k = \max\{|C|, |S|\}$, then*

$$\chi_h(G) = \begin{cases} |R| + k + 1, & \text{if } G \text{ is thin,} \\ n, & \text{otherwise.} \end{cases}$$

Lemma 3.6 (χ_c for union, join and quasi-spiders)**.** *Let G_1 and G_2 be two graphs. Then, $\chi_c(G_1 \cup G_2) = \max\{\chi_c(G_1), \chi_c(G_2)\}$ and $\chi_c(G_1 \vee G_2) = 2$. If G is a quasi-spider, then $\chi_c(G) = 2$.*

4. Coloring $(q, q-4)$-graphs

In this section, suppose that G is a $(q, q-4)$-graph which contains a separable p-component H, with bipartition (H_1, H_2) and at most q vertices, such that every vertex from $G - H$ is adjacent to all vertices in H_1 and non-adjacent to all vertices in H_2. Let n' be the number of vertices of $G - H$. If $G - H$ is empty, consider $\chi_a(G - H) = \chi_{st}(G - H) = \pi(G - H) = 0$. Given a coloring ψ of H, let $k(\psi)$ be the number of colors of ψ.

Theorems below prove that determining the chromatic numbers χ_a, χ_{st}, χ_h and χ_c for item (c) of Theorem 2.1 is linear time solvable, if q is a fixed integer.

Lemma 4.1. *Given a coloring ψ of H, let $k_2(\psi)$ be the number of colors with no vertex of H_1 and with no vertex of H_2 which is neighbor of two vertices from H_1 with the same color. Then*

$$\begin{aligned}
\chi_a(G) &= \min\Big\{ \min_{\psi\in C_a(H)} \Big\{k(\psi)+\max\{0,n'-k_2(\psi)\}\Big\}, \\
&\qquad \min_{\psi'\in C'_a(H)} \Big\{k(\psi')+\max\{0,\chi_a(G-H)-k_2(\psi')\}\Big\}\Big\} \\
\chi_{st}(G) &= \min\Big\{ \min_{\psi\in C_{st}(H)} \Big\{k(\psi)+\max\{0,n'-k_2(\psi)\}\Big\}, \\
&\qquad \min_{\psi'\in C'_{st}(H)} \Big\{k(\psi')+\max\{0,\chi_{st}(G-H)-k_2(\psi')\}\Big\}\Big\} \\
\pi(G) &= \min\Big\{ \min_{\psi\in C_\pi(H)} \Big\{k(\psi)+\max\{0,n'-k_2(\psi)\}\Big\}, \\
&\qquad \min_{\psi'\in C'_\pi(H)} \Big\{k(\psi')+\max\{0,\pi(G-H)-k_2(\psi')\}\Big\}\Big\}
\end{aligned}$$

where $C_a(H)$, $C_{st}(H)$ and $C_\pi(H)$ are respectively the set of all acyclic, star and nonrepetitive colorings of H, and $C'_a(H)\subseteq C_a(H)$, $C'_{st}(H)\subseteq C_{st}(H)$ and $C'_\pi(H)\subseteq C_\pi(H)$ are respectively the subsets of acyclic, star and nonrepetitive colorings such that all vertices from H_1 receive distinct colors.

Lemma 4.2. *Let $C_h(H)$ be the set of all harmonious colorings of H such that all vertices of H_1 have distinct colors. Then*

$$\chi_h(G) \;=\; n' + \min_{\psi\in C_h(H)} \Big\{k(\psi)\Big\}$$

Lemma 4.3. *If $G-H$ is not empty, then $\chi_c(G)=2$ (coloring the vertices of $G-H$ and H_2 with the color 1 and the vertices of H_1 with the color 2). If $G-H$ is empty, then G has less than q vertices and*

$$\chi_c(G) \;=\; \min_{\psi\in C_c(H)} \Big\{k(\psi)\Big\},$$

where $C_c(H)$ is the set of all clique-colorings of H.

Theorem 4.4. *If G is a P_4-tidy or $(q,q-4)$-graph, then we can obtain a minimum acyclic-star-harmonious-clique coloring of G and determine $\chi_a(G)$, $\chi_{st}(G)$, $\chi_h(G)$ and $\chi_c(G)$ in linear time.*

Proof. From Section 2, we can obtain the primeval decomposition in linear time. From lemmas of Sections 3 and 4, we are finished. □

5. Technical proofs

We now provide the proofs of the most important results of the paper. Firstly, we need to state a definition and recall a theorem from [8].

Definition 5.1. *Let $G=(V,E)$ be a graph. A subset M of V with $1\le |M|\le |V|$ is called a* module *if each vertex in $V-M$ is either adjacent to all vertices of M or to none of them. A module M is called a* homogeneous set *if $1<|M|<|V|$. The graph obtained from G by shrinking every maximal homogeneous set to one single vertex is called the* characteristic graph *of G.*

A graph is called *split graph* if its vertex set has a partition (K,S) such that K induces a clique and S induces an independent set.

Lemma 5.2 ([8]). *A p-connected graph G is separable if and only if its characteristic graph is a split graph.*

The rest of the paper is dedicated to prove Theorem 4.4 and lemmas from Sections 2 and 3.

5.1. **Acyclic, star and nonrepetitive colorings.** We start with the proofs of Lemmas 3.2, 3.3 and 3.4.

Proof of Lemma 3.2. If $G = G_1 \cup G_2$, then every color of G_1 can be used in G_2, and vice-versa. Thus, $\pi(G) = \max\{\pi(G_1), \pi(G_2)\}$. So, let $G = G_1 \vee G_2$. Suppose that $|V(G_1)| \geq 2$ and $|V(G)| \geq 2$. Let $a_1, b_1 \in V(G_1)$ and $a_2, b_2 \in V(G_2)$. Suppose that a_1 and b_1 receive color C_1 and that a_2 and b_2 receive color C_2. Then we have the bicolored P_4 $a_1a_2b_1b_2$, which is the repetition pattern $C_1C_2C_1C_2$; a contradiction. So, (a) all vertices of G_1 have distinct colors; or (b) all vertices of G_2 have distinct colors. □

Proof of Lemma 3.3. Let G be a spider with partition (R, C, S), such that $|C| = |S| = k$. A minimum acyclic coloring of G can be easily obtained from an acyclic coloring of $G[R]$, by assigning a new color for each vertex in C and finally by coloring each vertex of S with any appropriated avaiable color of C. Thus, $\chi_a(G) = \chi_a(G[R]) + k$.

On the other hand, to produce a star coloring of G, we first color optimally $G[R]$ and then assign one new color to each vertex of C. If G is thin, we color each vertex of S with any appropriated available color of C. If G is thick and $R \neq \emptyset$, then we use one of the colors of R to color every vertex of S. Then, $\chi_{st}(G) = \chi_{st}(R) + k$. If G is thick and $R = \emptyset$, then we have to add a new color and assign it to every vertex of S. By consequence, $\chi_{st}(G) = k + 1$. The same arguments can be used to $\pi(G)$. □

Proof of Lemma 3.4. Let G be a quasi-spider with partition (R, C, S) and $\min\{|C|, |S|\} = k$ and $\max\{|C|, |S|\} = k+1$. Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$ in the definition of quasi-spider. As before, we obtain a minimum acyclic(star) coloring of G by an acyclic(star) coloring R and associating a new color to each vertex of C. If $H = K_2$ and $H \in C$, then we need $k + 1$ colors to color the vertices of C. If $H = \overline{K_2}$, $H \in C$ and we use only one vertex to color H, then we have to give a new color to at least one vertex of S, otherwise we would have a bichromatic C_4 with H, this vertex of S and a vertex of R or C.

Then, if $H \in C$, then it is better to use $k + 1$ colors on C (independently if $H = K_2$ or $H = \overline{K_2}$). We will assume this. If $H \notin C$, then we use k colors on C. Now, we have to color the vertices of S.

At first, if $H \in C$, we can color each vertex of S using a color of some vertex in C, without producing any bichromatic cycle (but we can produce a bichromatic P_4). Then, if $H \in C$, we need no further new colors to acyclic color S.

If $H \in S$ and G is thin, we can always use a color of C to color each vertex of S without producing a bichromatic P_4. If $H \in S$, G is thick and $R \neq \emptyset$, we can use one color of R to color all vertices of S without producing a bichromatic P_4. In these cases, we do not need further new colors to color S.

If $H = K_2 \in S$, G is thick and $R = \emptyset$, we have create a new color and give it to all vertices of S (except one vertex of $H = K_2 \in S$, whose color could be some color used in C), without producing a bichromatic P_4. In this case, we need only one new color. If $H = \overline{K_2}$, we can use a color of C.

If $H \in C$, G is thick and $R = \emptyset$, we have to create a new color and give it to all vertices of S. In this case, we need only one new color. And we have finished all possibilities. The same arguments can be used to $\pi(G)$. □

Proof of Lemma 4.1. Clearly, an acyclic-star coloring of G induces an acyclic-star coloring of $G-H$. Notice that if $x \in G-H$ and $v \in H_2$ have the same color and if v has two neighbors u_1 and u_2 with the same color, then we have a bichromatic C_4 $(x-u_1-v-u_2)$. So, this cannot happen in an acyclic coloring of G.

Also observe that, in an acyclic coloring of G, there not exist vertices $x, y \in G-H$ with the same color and vertices $u, v \in H_1$ with the same color (otherwise, we would have a bichromatic C_4 $x-u-y-v$). Therefore, we have two options: (a) each vertex of $G-H$ receives a distinct color or (b) each vertex of H_1 receives a distinct color. Firstly, assume (b).

Observe that there is no bichromatic P_4 with three vertices of $G-H$ and a vertex of H_1, since every vertex of $G-H$ is adjacent to every vertex of H_1. Furthermore, there is no bichromatic P_4 with one vertex of $G-H$ and two vertices of H_1, since from (b) these three vertices must have distinct colors.

As a conclusion, there is no bichromatic induced cycle with a vertex of $G-H$ and a vertex of H_1. Thus every minimum acyclic coloring of G which satisfies (b) induces a minimum acyclic coloring of $G-H$.

By Lemma 5.2, if $G-H$ is not empty, then the characteristic graph of H is a split graph (H_1 represents the clique and H_2 represents the independent set). Then, there is no induced P_3 with one vertex u of H_1 and two vertices w_1, w_2 of H_2, since w_1 and w_2 must be in the same maximal homogeneous set, and therefore u is adjacent to both or none of w_1 and w_2. By consequence, there is also no bichromatic induced P_4 with one vertex of $G-H$, one vertex u of H_1 and two vertices w_1, w_2 of H_2 (otherwise we would have an induced P_3 $u-w_1-w_2$). Therefore, there is no bichromatic induced P_4 with a vertex of $G-H$ and a vertex of H_1. As a consequence, every minimum star coloring of G which satisfies (b) induces a minimum star coloring of $G-H$.

Therefore, our work is to obtain a minimum acyclic (star) coloring of G satisfying (b), given a minimum acyclic (star) coloring of $G-H$. Let ψ'_a and ψ'_s be respectively an acyclic (star) coloring of H such that the vertices of H_1 receive distinct colors.

Then we get respectively $k(\psi'_a) + \max\{0, \chi_a(G-H) - k_2(\psi'_a)\}$ and $k(\psi'_s) + \max\{0, \chi_{st}(G-H) - k_2(\psi'_s)\}$ colors for the acyclic-star colorings of G respecting ψ'_a and ψ'_s. Since H has at most q vertices, we can search in constant time the colorings ψ'_a and ψ'_s that minimize those values.

Now, assume (a). This case is easier than (b). Let ψ_a and ψ_s be respectively an acyclic-star coloring of H (with no further restrictions). Then we get respectively $k(\psi_a) + \max\{0, |V(G-H)| - k_2(\psi_a)\}$ and $k(\psi_s) + \max\{0, |V(G-H)| - k_2(\psi_s)\}$ colors for the acyclic-star colorings of G respecting (a), ψ_a and ψ_s. Since H has at most q vertices, we can search in constant time the colorings ψ_a and ψ_s that minimize those values.

Finally, take the minimum between (a) and (b) to calculate χ_a and χ_{st}.

To calculate $\pi(G)$, we just have to prove that every star coloring of G has no repetition pattern with a vertex of $G-H$ and a vertex of H. To do this, consider by contradiction a star coloring of G with a color repetition pattern $x_1 \dots x_p x_1 \dots x_p$ on vertices $v_1 \dots v_{2p}$ with vertices of $G-H$ and H. Without loss of generality, consider that $v_1 \in G-H$. Let $v_k v_{k+1}$ be the first edge from $G-H$ to H_1. Clearly $k < p$ (otherwise, v_{k+1-p} and v_{k+1} have the same color x_{k+1-p} and induce an edge, a contradiction).

Observe that v_k and v_{k+p} received color x_k, and v_{k+p} and v_{k+p+1} received color x_{k+1}. If $v_{k+p} \in H_1$, then we have the edge $v_k v_{k+p}$ with colors x_k and x_k, a contradiction. If $v_{k+p+1} \in G-H$, then we have the edge $v_{k+1} v_{k+p+1}$ with colors x_{k+1} and x_{k+1}, a contradiction. If $v_{k+p} \in G-H$ and $v_{k+p+1} \in H_1$, then we have the bichromatic P_4 $v_k v_{k+1} v_{k+p} v_{k+p+1}$, a contradiction. If $v_{k+p} \in H_2$ and $v_{k+p+1} \in H_1$, then, from Lemma 5.2, v_{k+1} and v_{k+p+1} are in the same maximal homogeneous set and, consequently, $v_{k+1} v_{k+p}$ is an edge and we have the bichromatic P_4 $v_k v_{k+1} v_{k+p} v_{k+p+1}$, a contradiction.

Finally, the last case we have to consider is when v_{k+p} and v_{k+p+1} are in H_2. Observe that $v_{p+1}, \dots, v_{k+p+1} \in H_2$ (otherwise, we have some edge $v_\ell v_{p+\ell}$, $\ell \in \{1, \dots, k\}$, whose vertices have the same color x_ℓ, a contradiction) and are in the same maximal homogeneous set $M_2 \subseteq H_2$ from Lemma 5.2.

Let $\ell \in \{1, \dots, p\}$ be the minimum integer such that $v_{p+\ell} \notin M_2$ (this integer must exist, otherwise M_2 contains all colors and we have some edge from H_1 to M_2 whose vertices have the same color, a contradiction). Let $M_1 \subseteq H_1$ be the maximal homogeneous set which contains $v_{p+\ell}$. If $v_\ell \in G - H$, then we have the edge $v_\ell v_{p+\ell}$ whose vertices have the same color x_ℓ, a contradiction. If $v_\ell \in H_1$, then $v_\ell v_{p+\ell}$ could not be an edge and, from Lemma 5.2, $v_\ell \in M_1$ and, consequently, we have the edge $v_\ell v_{p+\ell-1}$ and the bichromatic P_4 $v_{\ell-1}v_\ell v_{p+\ell-1}v_{p+\ell}$, a contradiction.

If $v_\ell \in H_2$, let $t < \ell$ be the maximum integer not in H_2 smaller than ℓ. Clearly t exists and $v_{t+1}, \dots, v_\ell \in H_2$ and are in the same maximal homogenoeus set $M_2' \subseteq H_2$ from Lemma 5.2. If $v_t \in M_1$, then we have the edge $v_t v_{p+t}$ whose vertices have the same color x_t, a contradiction. If $v_t \notin M_1$, then, from Lemma 5.2, $v_t v_{p+\ell}$ is an edge and we have the bichromatic P_4 $v_\ell v_t v_{p+\ell} v_{p+t}$, a contradiction.

□

5.2. **Harmonious coloring.**

Proof of Lemma 3.5. Consider a harmonious coloring C of G. If G is the join of two graphs G_1 and G_2, then there is no two vertices x and y of G_1 with the same color, since they have a common neighbor in G_2. Similarly, we have the same for G_2. Since no color of G_1 can appear in G_2, we have that all vertices must have distinct colors in C.

Now consider G a spider with partition (S, C, R). Consider an harmonious coloring C of G. Since C induces a clique, all vertices of C must receive different colors. Since every vertex of R is adjacent to every vertex of C, then no color of C can occur in R (and vice-versa). If $|R| > 1$, then two vertices of R cannot have the same color, because they are adjacent to the same vertex in C (otherwise, the coloring is not harmonious). Summarizing, no two vertices of $R \cup C$ are assigned to the same color.

Consider now that G is a thin spider. Let s_i be a vertex of S. Let c_i be the vertex of C that is adjacent to s_i. The color of s_i cannot be the color of c_j of C, otherwise (c_i, s_i) and (c_i, c_j) would be edges with the same pair of color in their endpoints. For the same reason, the color of s_i cannot be the color of any vertex r_j of R. So, a new color, distinct from those used to color $R \cup C$, have to be used to color S. Then, $h(G) \leq |R| + |C| + 1$.

Observe that by coloring every vertex of S with this new color would not produce two edges with the same pair of colors in their endpoints, because this new color appears only in S and there is a bijection between vertices of S and C, i.e., each vertex of S is connected to exactly one vertex of C different from the others. Therefore, $h(G) \geq |R| + |C| + 1$.

Consider now that G is a thick spider with $|C| > 2$. Again, as before, the colors used in $C \cup R$ cannot be reused to color the vertices of S. It remains to know if two vertices s_i and s_j of S can receive the same color. As $|C| > 2$, there is a vertex c_k of C that is adjacent to s_i and s_j. Then, s_i and s_j must receive different colors on a harmonious coloring. It means that all the vertices of S have to be assigned to distinct colors, that is, $h(G) = |R| + |C| + |S|$. As $|S| = |C|$, we have the result. □

Proof of Lemma 4.2. Let c be an harmonious coloring of G. Let c_H be the restriction of c to H. Observe that as c is harmonious, then c_H is harmonious too. Therefore, every harmonious coloring of G can be obtained from some harmonious coloring of $G[H]$.

Note that there is no two vertices x and y of $G - H$ with the same color in c, since x and y have a common neighbor in H_1. Hence, all vertices of $G - H$ receive distinct colors.

Consider then a harmonious coloring c_H of $G[H]$. We are looking for a harmonious coloring c_G of G that extends c_H and uses a minimum number of colors. It is necessary to know which colors of c_H can be used to color the vertices in $G - H$. As all the vertices in $G - H$ are adjacent to all the vertices of H_1, then the colors that can be used are the ones in c_H such that no vertex with this color is adjacent to other vertex colored with some color that appears in H_1.

Summarizing, let c_1 be the set of the colors in c_H of the vertices in H_1. Let X be the subset of vertices in H colored with a color of c_1. Clearly, $H_1 \subseteq X$. Let $Y = X \cup N(X)$, where $N(X)$ is the set of the neighbors of the vertices in X. Let c_Y be the set of colors in c_H with a vertex of Y. Let c_Z be the set of colors in c_H that are not used in Y. That is, $c_Y \cup c_Z = c_H$ and $c_Y \cap c_Z = \emptyset$.

Clearly, we cannot use any color γ from c_Y in $G - H$, because either γ is a color of H_1 or it exists already a vertex of H with the color γ which is neighbor from some vertex in H_1. It is easy to see that we can use the colors of c_Z in $G - H$. With this, a harmonious coloring c_G of G that extends c_H and use a smallest number of colors must use $|c_H| + \max\{|V(G)| - |H| - |c_Z|, 0\}$ colors and it is obtained by coloring each vertex of $G - H$ either using colors of c_Z or adding new colors with respect to the colors used in c_H.

As the size of H is less than q (which is a constant independent from the size of G), we can get in constant time all the harmonious colorings c_H of H and calculate the minimum number of colors in a harmonious coloring of G that extends c_H. By doing this, we get in constant time the harmonious chromatic number of G and we can obtain on linear time in the number of vertices a harmonious coloring of G. □

5.3. **Clique coloring.**

Proof of Lemma 3.6. The proof is direct if $G = G_1 \cup G_2$. If G is the join of two graphs G_1 and G_2, then it is easy to see that every maximal clique of G must have vertices of G_1 and G_2, since, for every clique C of G_1, $C \cup \{v_2\}$ (where $v_2 \in G_2$) is a clique of G. Then, coloring the vertices of G_1 with color 1 and the vertices of G_2 with color 2, we have that every maximal clique receives two colors.

Now suppose that G is a quasi-spider with partition (R, C, S). Suppose first that R is not empty. The same argument below shows that there is no maximal clique of G with vertices of R and no vertex of C and that there is no maximal clique of G with vertices of C and no vertex of R or no vertex of S. Since there is no clique with two vertices of S, we can obtain a clique coloring of G by coloring the vertices of R and S with color 1 and the vertices of C with color 2.

Suppose now that R is empty. In this case, is is possible that C is a maximal clique. Let $H = K_2$ or $H = \overline{K_2}$ be the subgraph that replaced a vertex of $C \cup S$ in the definition of quasi-spider. Let $x \in C - H$ be a vertex of C that is not in H. Let $N(x)$ be the set of neighbors of x in S. It is easy to see that coloring $C - \{x\}$ and $N(x)$ with color 1 and x and $S - N(x)$ with color 2, we have that every maximal clique receives two colors. Then we have a 2-clique-coloring of G. □

Proof of Lemma 4.3. At first, suppose that $G - H$ is not empty. Then H is a separable p-component and, by Lemma 5.2, the characteristic graph of H is a split graph (H_1 "*reduces*" to a clique and H_2 "*reduces*" to an independente set). Also remember that every vertex of H_2 has a neighbor in H_1. Hence, if two vertices of H_2 induces an edge, then they are in the same homogeneous set and then they have a common neighbor in H_1. Consequently, there is no maximal clique with vertex set contained in H_2.

It is easy to see that there is no maximal clique of G with vertices of $G - H$ and no vertex of H_1, since, for every clique C of $G - H$, $C \cup \{v\}$ (where $v \in H_1$) is a clique of G. The same argument shows that there is no maximal clique of G with vertices of H_1 and no vertex of H_2

or no vertex of $G-H$.Then, we can obtain a clique coloring of G by coloring the vertices of $G-H$ and H_2 with color 1 and the vertices of H_1 with color 2.

Now, suppose that $G-H$ is empty. Since H is a p-connected $(q, q-4)$-graph, then H has at most $q-1$ vertices. Since q is fixed, we can generate all possible clique-colorings in constant time and obtain the clique chromatic number. □

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